

CHAPTER 2

THE DISCRETE WAVELET TRANSFORM

2.1 Introduction

The transform of a signal is just another form of representing the signal. It does not change the information content present in the signal. The Wavelet Transform provides a time-frequency representation of the signal. It was developed to overcome the short coming of the Short Time Fourier Transform (STFT), which can also be used to analyze non-stationary signals. While STFT gives a constant resolution at all frequencies, the Wavelet Transform uses multi-resolution technique by which different frequencies are analyzed with different resolutions.

A wave is an oscillating function of time or space and is periodic. In contrast, wavelets are localized waves. They have their energy concentrated in time or space and are suited to analysis of transient signals. While Fourier Transform and STFT use waves to analyze signals, the Wavelet Transform uses wavelets of finite energy.

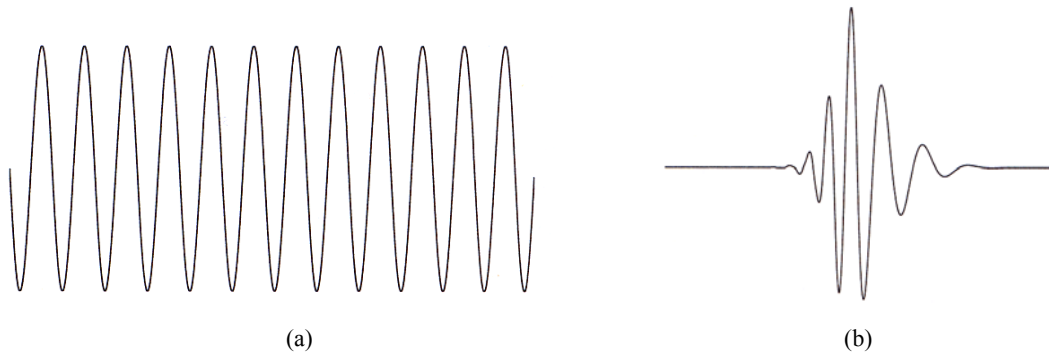


Figure 2.1 Demonstration of (a) a Wave and (b) a Wavelet [2].

The wavelet analysis is done similar to the STFT analysis. The signal to be analyzed is multiplied with a wavelet function just as it is multiplied with a window function in STFT, and then the transform is computed for each segment generated. However, unlike STFT, in Wavelet Transform, the width of the wavelet function changes with each spectral component. The Wavelet Transform, at high frequencies, gives good time resolution and poor frequency resolution, while at low frequencies, the Wavelet Transform gives good frequency resolution and poor time resolution.

2.2 The Continuous Wavelet Transform and the Wavelet Series

The Continuous Wavelet Transform (CWT) is provided by equation 2.1, where $x(t)$ is the signal to be analyzed. $\psi(t)$ is the mother wavelet or the basis function. All the wavelet functions used in the transformation are derived from the mother wavelet through translation (shifting) and scaling (dilation or compression).

$$X_{WT}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \cdot \psi^* \left(\frac{t - \tau}{s} \right) dt \quad 2.1$$

The mother wavelet used to generate all the basis functions is designed based on some desired characteristics associated with that function. The translation parameter τ relates to the location of the wavelet function as it is shifted through the signal. Thus, it corresponds to the time information in the Wavelet Transform. The scale parameter s is defined as $|1/\text{frequency}|$ and corresponds to frequency information. Scaling either dilates (expands) or compresses a signal. Large scales (low frequencies) dilate the signal and provide detailed information hidden in the signal, while small scales (high frequencies) compress the signal and provide global information about the signal. Notice that the Wavelet Transform merely performs the convolution operation of the signal and the basis function. The above analysis becomes very useful as in most practical applications, high frequencies (low scales) do not last for a long duration, but instead, appear as short bursts, while low frequencies (high scales) usually last for entire duration of the signal.

The Wavelet Series is obtained by discretizing CWT. This aids in computation of CWT using computers and is obtained by sampling the time-scale plane. The sampling rate can be changed accordingly with scale change without violating the Nyquist criterion. Nyquist criterion states that, the minimum sampling rate that allows reconstruction of the original signal is 2ω radians, where ω is the highest frequency in the signal. Therefore, as the scale goes higher (lower frequencies), the sampling rate can be decreased thus reducing the number of computations.

2.3 The Discrete Wavelet Transform

The Wavelet Series is just a sampled version of CWT and its computation may consume significant amount of time and resources, depending on the resolution required. The Discrete Wavelet Transform (DWT), which is based on sub-band coding is found to yield a fast computation of Wavelet Transform. It is easy to implement and reduces the computation time and resources required.

The foundations of DWT go back to 1976 when techniques to decompose discrete time signals were devised [5]. Similar work was done in speech signal coding which was named as sub-band coding. In 1983, a technique similar to sub-band coding was developed which was named pyramidal coding. Later many improvements were made to these coding schemes which resulted in efficient multi-resolution analysis schemes.

In CWT, the signals are analyzed using a set of basis functions which relate to each other by simple scaling and translation. In the case of DWT, a time-scale representation of the digital signal is obtained using digital filtering techniques. The signal to be analyzed is passed through filters with different cutoff frequencies at different scales.

2.4 DWT and Filter Banks

2.4.1 Multi-Resolution Analysis using Filter Banks

Filters are one of the most widely used signal processing functions. Wavelets can be realized by iteration of filters with rescaling. The resolution of the signal, which is a measure of the amount of detail information in the signal, is determined by the filtering operations, and the scale is determined by upsampling and downsampling (subsampling) operations[5].

The DWT is computed by successive lowpass and highpass filtering of the discrete time-domain signal as shown in figure 2.2. This is called the Mallat algorithm or Mallat-tree decomposition. Its significance is in the manner it connects the continuous-time multiresolution to discrete-time filters. In the figure, the signal is denoted by the sequence $x[n]$, where n is an integer. The low pass filter is denoted by G_0 while the high pass filter is denoted by H_0 . At each level, the high pass filter produces detail information, $d[n]$, while the low pass filter associated with scaling function produces coarse approximations, $a[n]$.

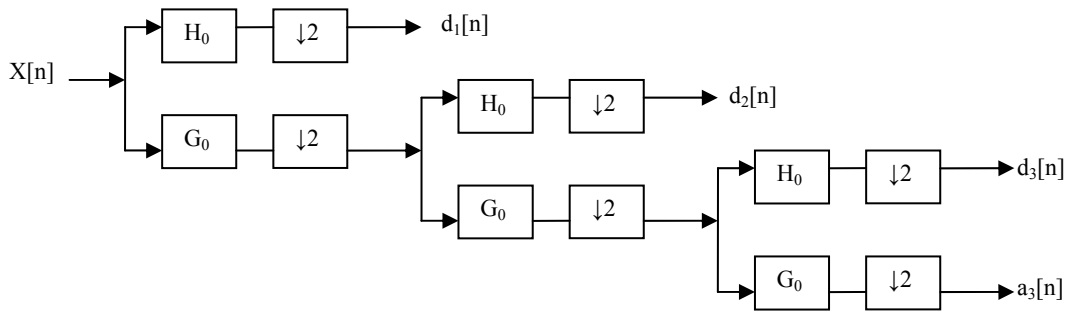


Figure 2.2 Three-level wavelet decomposition tree.

At each decomposition level, the half band filters produce signals spanning only half the frequency band. This doubles the frequency resolution as the uncertainty in frequency is reduced by half. In accordance with Nyquist's rule if the original signal has

a highest frequency of ω , which requires a sampling frequency of 2ω radians, then it now has a highest frequency of $\omega/2$ radians. It can now be sampled at a frequency of ω radians thus discarding half the samples with no loss of information. This decimation by 2 halves the time resolution as the entire signal is now represented by only half the number of samples. Thus, while the half band low pass filtering removes half of the frequencies and thus halves the resolution, the decimation by 2 doubles the scale.

With this approach, the time resolution becomes arbitrarily good at high frequencies, while the frequency resolution becomes arbitrarily good at low frequencies. The time-frequency plane is thus resolved as shown in figure 1.1(d) of Chapter 1. The filtering and decimation process is continued until the desired level is reached. The maximum number of levels depends on the length of the signal. The DWT of the original signal is then obtained by concatenating all the coefficients, $a[n]$ and $d[n]$, starting from the last level of decomposition.

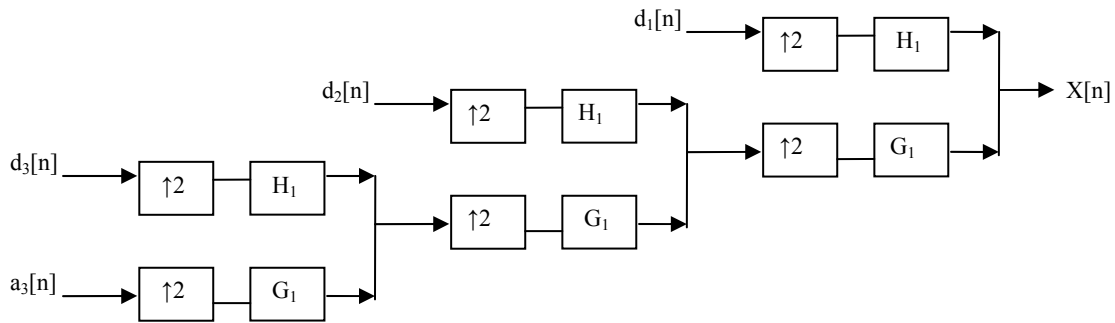


Figure 2.3 Three-level wavelet reconstruction tree.

Figure 2.3 shows the reconstruction of the original signal from the wavelet coefficients. Basically, the reconstruction is the reverse process of decomposition. The approximation and detail coefficients at every level are upsampled by two, passed through the low pass and high pass synthesis filters and then added. This process is continued through the same number of levels as in the decomposition process to obtain

the original signal. The Mallat algorithm works equally well if the analysis filters, G_0 and H_0 , are exchanged with the synthesis filters, G_1 and H_1 .

2.4.2 Conditions for Perfect Reconstruction

In most Wavelet Transform applications, it is required that the original signal be synthesized from the wavelet coefficients. To achieve perfect reconstruction the analysis and synthesis filters have to satisfy certain conditions. Let $G_0(z)$ and $G_1(z)$ be the low pass analysis and synthesis filters, respectively and $H_0(z)$ and $H_1(z)$ the high pass analysis and synthesis filters respectively. Then the filters have to satisfy the following two conditions as given in [4] :

$$G_0(-z) G_1(z) + H_0(-z) H_1(z) = 0 \quad 2.2$$

$$G_0(z) G_1(z) + H_0(z) H_1(z) = 2z^{-d} \quad 2.3$$

The first condition implies that the reconstruction is aliasing-free and the second condition implies that the amplitude distortion has amplitude of one. It can be observed that the perfect reconstruction condition does not change if we switch the analysis and synthesis filters.

There are a number of filters which satisfy these conditions. But not all of them give accurate Wavelet Transforms, especially when the filter coefficients are quantized. The accuracy of the Wavelet Transform can be determined after reconstruction by calculating the Signal to Noise Ratio (SNR) of the signal. Some applications like pattern recognition do not need reconstruction, and in such applications, the above conditions need not apply.

2.4.3 Classification of wavelets

We can classify wavelets into two classes: (a) orthogonal and (b) biorthogonal. Based on the application, either of them can be used.

(a)Features of orthogonal wavelet filter banks

The coefficients of orthogonal filters are real numbers. The filters are of the same length and are not symmetric. The low pass filter, G_0 and the high pass filter, H_0 are related to each other by

$$H_0(z) = z^{-N} G_0(-z^{-1}) \quad 2.4$$

The two filters are alternated flip of each other. The alternating flip automatically gives double-shift orthogonality between the lowpass and highpass filters [1], i.e., the scalar product of the filters, for a shift by two is zero. i.e., $\sum G[k] H[k-2l] = 0$, where $k, l \in \mathbb{Z}$ [4]. Filters that satisfy equation 2.4 are known as Conjugate Mirror Filters (CMF). Perfect reconstruction is possible with alternating flip.

Also, for perfect reconstruction, the synthesis filters are identical to the analysis filters except for a time reversal. Orthogonal filters offer a high number of vanishing moments. This property is useful in many signal and image processing applications. They have regular structure which leads to easy implementation and scalable architecture.

(b)Features of biorthogonal wavelet filter banks

In the case of the biorthogonal wavelet filters, the low pass and the high pass filters do not have the same length. The low pass filter is always symmetric, while the high pass filter could be either symmetric or anti-symmetric. The coefficients of the filters are either real numbers or integers.

For perfect reconstruction, biorthogonal filter bank has all odd length or all even length filters. The two analysis filters can be symmetric with odd length or one symmetric and the other antisymmetric with even length. Also, the two sets of analysis and synthesis filters must be dual. The linear phase biorthogonal filters are the most popular filters for data compression applications.

2.5 Wavelet Families

There are a number of basis functions that can be used as the mother wavelet for Wavelet Transformation. Since the mother wavelet produces all wavelet functions used in the transformation through translation and scaling, it determines the characteristics of the resulting Wavelet Transform. Therefore, the details of the particular application should be taken into account and the appropriate mother wavelet should be chosen in order to use the Wavelet Transform effectively.

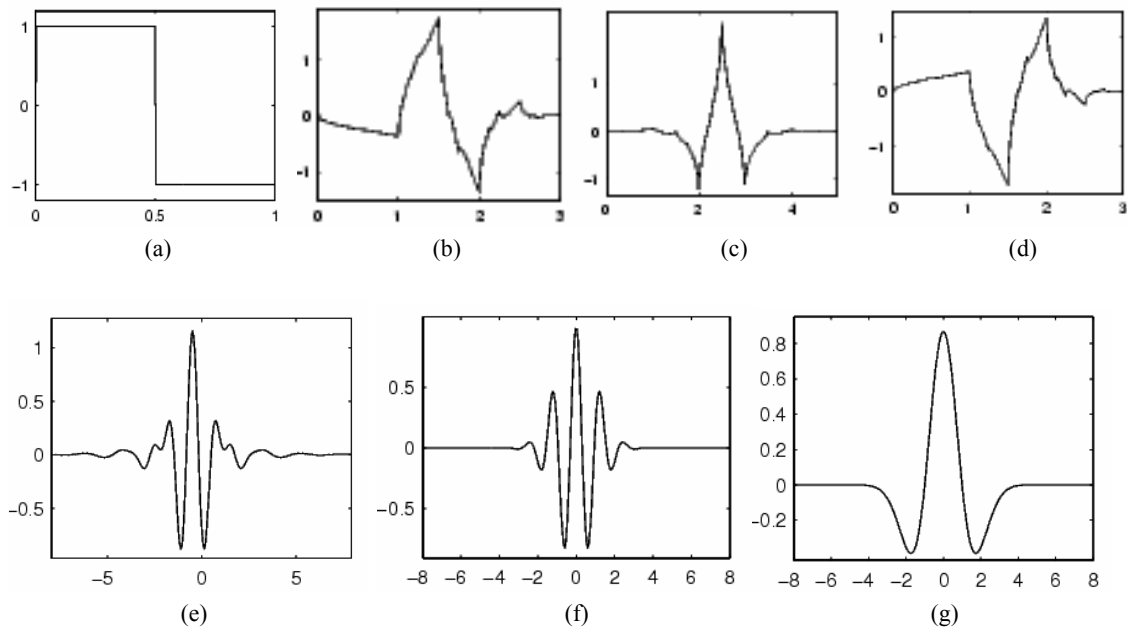


Figure 2.4 Wavelet families (a) Haar (b) Daubechies4 (c) Coiflet1 (d) Symlet2 (e) Meyer (f) Morlet (g) Mexican Hat.

Figure 2.4 illustrates some of the commonly used wavelet functions. Haar wavelet is one of the oldest and simplest wavelet. Therefore, any discussion of wavelets starts with the Haar wavelet. Daubechies wavelets are the most popular wavelets. They represent the foundations of wavelet signal processing and are used in numerous applications. These are also called Maxflat wavelets as their frequency responses have maximum flatness at frequencies 0 and π . This is a very desirable property in some

applications. The Haar, Daubechies, Symlets and Coiflets are compactly supported orthogonal wavelets. These wavelets along with Meyer wavelets are capable of perfect reconstruction. The Meyer, Morlet and Mexican Hat wavelets are symmetric in shape. The wavelets are chosen based on their shape and their ability to analyze the signal in a particular application.

2.6 Applications

There is a wide range of applications for Wavelet Transforms. They are applied in different fields ranging from signal processing to biometrics, and the list is still growing. One of the prominent applications is in the FBI fingerprint compression standard. Wavelet Transforms are used to compress the fingerprint pictures for storage in their data bank. The previously chosen Discrete Cosine Transform (DCT) did not perform well at high compression ratios. It produced severe blocking effects which made it impossible to follow the ridge lines in the fingerprints after reconstruction. This did not happen with Wavelet Transform due to its property of retaining the details present in the data.

In DWT, the most prominent information in the signal appears in high amplitudes and the less prominent information appears in very low amplitudes. Data compression can be achieved by discarding these low amplitudes. The wavelet transforms enables high compression ratios with good quality of reconstruction. At present, the application of wavelets for image compression is one the hottest areas of research. Recently, the Wavelet Transforms have been chosen for the JPEG 2000 compression standard.

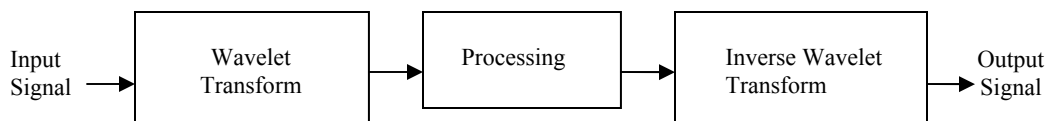


Figure 2.5 Signal processing application using Wavelet Transform.

Figure 2.5 shows the general steps followed in a signal processing application. Processing may involve compression, encoding, denoising etc. The processed signal is either stored or transmitted. For most compression applications, processing involves quantization and entropy coding to yield a compressed image. During this process, all the wavelet coefficients that are below a chosen threshold are discarded. These discarded coefficients are replaced with zeros during reconstruction at the other end. To reconstruct the signal, the entropy coding is decoded, then quantized and then finally Inverse Wavelet Transformed.

Wavelets also find application in speech compression, which reduces transmission time in mobile applications. They are used in denoising, edge detection, feature extraction, speech recognition, echo cancellation and others. They are very promising for real time audio and video compression applications. Wavelets also have numerous applications in digital communications. Orthogonal Frequency Division Multiplexing (OFDM) is one of them. Wavelets are used in biomedical imaging. For example, the ECG signals, measured from the heart, are analyzed using wavelets or compressed for storage. The popularity of Wavelet Transform is growing because of its ability to reduce distortion in the reconstructed signal while retaining all the significant features present in the signal.